MA 362 Quiz 3 – Extensions of finite degree – Wednesday, Feb 6

1. Let F be a field and consider a field extension $F(\alpha, \beta)$ where $[F(\alpha) : F] = 3$ and $[F(\alpha, \beta) : F(\alpha)] = 2$

   (a) What is the degree $[F(\alpha, \beta) : F]$?

   $3 \cdot 2 = 6$

   (b) Use α and β to write down a basis for $F(\alpha, \beta)$ as an F-vector space.

   $\{1, \alpha, \alpha^2, \beta, \alpha\beta, \alpha^2\beta\}$

2. Let $F \subseteq E \subseteq K \subseteq L$ be fields, and suppose that $[L : F] = 42$, $[E : F] = 2$, and $[L : K] = 3$. What is $[K : E]$?

$$\begin{array}{c|c|c} & L & \\ & | & 3 \\ & K & \\ 42 & | & \\ & E & \\ & | & 2 \\ & F & \end{array}$$

so $2 \cdot [K : E] \cdot 3 = 42$

$\Rightarrow [K : E] = \frac{42}{6} = 7$

3. Suppose $F \subseteq E$ is a field extension and $[E : F]$ is finite. Explain why E is an *algebraic* extension of F. (You don't need to do a whole careful proof. Just explaining the key ideas is enough.)

Let $\alpha \in E$.

Then, if $[E:F] = n$, we know the set $\{1, \alpha, \alpha^2, \dots, \alpha^n\}$ is linearly <u>dependent</u> (since it has $n+1$ elements).

So $\exists$ a nontrivial expression

$$\sum_{i=0}^{n} a_i \alpha^i = 0 \quad \text{for } a_i \in F.$$

So $$f(x) = \sum_{i=0}^{n} a_i x^i \in F[x]$$

has α as a root.