MA 362 — Homework 5 — Due Friday, Feb 22

1. Let $F \subseteq E$ be an algebraic field extension. Prove that $\overline{F}$ and $\overline{E}$ are isomorphic.

2. Consider the polynomial $x^3 - 2 \in \mathbb{Q}[x]$. You can check that in the field $\mathbb{C}$, the roots of this polynomial are:

$$\beta_1 = \sqrt[3]{2}, \qquad \beta_2 = \sqrt[3]{2}\left(\frac{-1+i\sqrt{3}}{2}\right), \qquad \beta_3 = \sqrt[3]{2}\left(\frac{-1-i\sqrt{3}}{2}\right)$$

   a. Let $E = \mathbb{Q}(\beta_1, \beta_2, \beta_3)$. Prove that $E = \mathbb{Q}\left(\sqrt[3]{2}, \frac{-1+i\sqrt{3}}{2}\right) = \mathbb{Q}(\sqrt[3]{2}, i\sqrt{3})$, and explain why E is the *splitting field* of $x^3 - 2$ in $\mathbb{C}$.

   b. Consider the field $L = \mathbb{Q}[x]/(x^3 - 2)$. Prove that L is not a splitting field of $x^3 - 2$. It may help you to think about the degrees of E and L over $\mathbb{Q}$.

   c. By the isomorphism extension theorem, we know that there exists an isomorphism from L to a subfield $\mathbb{C}$ extending the identity may $\mathbb{Q} \hookrightarrow \mathbb{C}$. Describe the set of all such isomorphism extensions.

   d. Describe the set of all automorphisms from E to itself that fix the subfield $\mathbb{Q}$.

   e. Describe the set of all automorphisms from E to itself that fix the subfield $\mathbb{Q}(\sqrt[3]{2})$.

3. Find the degree of the splitting field of the given polynomials in $\mathbb{Q}[x]$.

   a. $x^3 - 1$

   b. $x^4 - 1$

   c. $(x^2 - 2)(x^3 - 2)$

4. Let $F \subseteq E$ be a field extension and assume that $[E : F]$ is finite, and that E is a splitting field over F. Prove that there exists a single polynomial $f(x) \in F[x]$ so that E is the splitting field of $f(x)$.

5. Let $F \subseteq E \subseteq \overline{F}$ be a splitting field contained in an algebraic closure of F, and let $f(x) \in F[x]$ be an irreducible polynomial. Prove that if $f(x)$ has a root it E, then $f(x)$ has all of its roots in E.

Bonus. Let $F \subseteq E$ be a splitting field with $[E : F] < \infty$, and let $G(E/F)$ be the group of all automorphisms of E that fix F. Prove that $G(E/F)$ is a *finite* group, and furthermore, prove that $|G(E/F)| \leqslant n!$, where n is the minimum degree of a polynomial $f(x)$ so that E is the splitting field of F (See problem 4 above).