Math 362 HW 1

1. a. Proceed by induction on degree:

if $f(x) = ax+b$ (degree 1), then $f(x)$ is irreducible, because for any $f(x) = g(x)h(x)$ we have $\deg(g(x)) + \deg(h(x)) = 1$, so we can't have $\deg(g(x)) = 0 = \deg(h(x))$. One of the factors must also have degree 1.

Inductive Step: Choose $f(x)$ of degree n. If $f(x)$ is irreducible already, then we're done. If not, then we can write $f(x) = g(x)h(x)$ with $\deg(g(x)) < n$, $\deg(h(x)) < n$.

By induction, both $g(x)$ and $h(x)$ can be written as products of irreducible polynomials. Multiplying these factorizations together gives $f(x)$ as a product of irreducible polynomials.

b. Suppose we have two factorizations

$$f(x) = p_1(x)\, p_2(x) \ldots p_n(x)$$
$$= q_1(x)\, q_2(x) \ldots q_m(x) \qquad \text{(WLOG: Assume } m \geq n)$$

Since $p_1(x) \mid q_1(x) q_2(x) \ldots q_m(x)$ we know

$p_1(x) \mid q_i(x)$ for some i, so

$q_i(x) = a(x) p_1(x)$ for some $a(x)$. But $q_i(x)$ is irreducible, so is $p_1(x)$, so $a(x)$ must be a constant polynomial, $a(x) = u$, $u \in F$, $u \neq 0$.

we can rearrange the $q_i(x)$ indexing so that

$p_1(x) = u \cdot q_1(x)$, which lets us rewrite and cancel,

so $$p_2(x) \cdots p_n(x) = \frac{1}{u_1} q_2(x) \cdots q_m(x).$$

Continuing in this way we eventually get to

$$1 = \frac{1}{u_1 u_2 \cdots u_n} q_{n+1}(x)\, q_{n+2}(x) \cdots q_m(x).$$

which is impossible, since $\deg(1) = 0$, $\deg(RHS) > 0$, unless $m = n$, in which case

$$p_1(x) = u_1\, q_1(x)$$
$$p_2(x) = u_2\, q_2(x)$$
$$\vdots$$
$$p_n(x) = u_n\, q_n(x).$$

2. $\alpha = \sqrt{11}$

$\alpha^2 = 11$

$\alpha^2 - 11 = 0 \Rightarrow f(x) = x^2 - 11.$

$\beta = \sqrt{3} + \sqrt{5}$

$\beta^2 = 3 + 2\sqrt{15} + 5$

$\dfrac{\beta^2 - 8}{2} = \sqrt{15}$

$\left(\dfrac{\beta^2 - 8}{2}\right)^2 - 15 = 0 \Rightarrow f(x) = \frac{1}{4}x^4 - 4x^2 + 1$

(this is irreducible)

c. $\gamma = \sqrt[3]{7}$

$\gamma^3 = 7$

$\gamma^3 - 7 = 0$ so $f(x) = x^3 - 7$.

(this is irreducible)

d. $\delta = \sqrt[3]{8}$

$\delta^3 - 8 = 0$ So $f(x) = x^3 - 8$.

But this has $x-2$ as a factor, it's not irreducible.

In fact, $\delta = 2$.

3. The answer to this question will be an irreducible factor of $x^6 - 1 = (x-1)(x+1)(x^2-x+1)(x^2+x+1)$.

To find out which one, plug in ζ to each and see when you get 0.

Clearly, $x-1$ and $x+1$ don't work.

$(\zeta^2 + \zeta + 1)$ simplifies to $1 + \sqrt{3}\, i$.

$(\zeta^2 - \zeta + 1)$ simplifies to 0. So

$f(x) = x^2 - x + 1$ is ~~the~~ an irred. poly that has ζ as a root.

4. $E = \dfrac{\mathbb{Q}[x]}{(x^2-2)}$ has basis

$\{1 + (x^2-2),\ x + (x^2-2)\}$.

Since $x^2 + (x^2-2) = 2 + (x^2-2)$ we can rewrite any element in E, say

$\sum_{i=0}^{n} a_i x^i + (x^2-2)$, as:

$$= \sum_{\substack{i=0 \\ i \text{ even}}}^{n} a_i 2^{i/2} + \sum_{\substack{i=0 \\ i \text{ odd}}}^{n} a_i 2^{\frac{i-1}{2}} x + (x^2-2)$$

$$= \left(\sum_{\substack{i=0 \\ i \text{ even}}}^{n} a_i 2^{i/2}\right)(1 + (x^2-2)) + \left(\sum_{\substack{i=0 \\ i \text{ odd}}}^{n} a_i 2^{\frac{i-1}{2}}\right)(x + (x^2-2))$$

so $\{1 + (x^2-2),\ x + (x^2-2)\}$ spans E over $\mathbb{Q}$.

it's a basis because if there were a spaning set of size 1, Then E would be a 1-dimensional $\mathbb{Q}$-vector space, so $E = \mathbb{Q}$, a contradiction (E contains a square root of 2, but $\mathbb{Q}$ doesn't).

5. in $\mathbb{R}$, any demnt α satisfies $\alpha^2 \geq 0$.

$$E = \frac{\mathbb{Q}[x]}{x^2+1}$$ is an algebraic extension of $\mathbb{Q}$,

and the element $\alpha = x + (x^2+1)$ satisfies that

$\alpha^2 = -1 + (x^2+1)$, the additive inverse of the identity.

$\mathbb{R}$ contains no elements with this property, so

E is not a subfield of $\mathbb{R}$.