MA 362 — Homework 10 — Due Friday, April 12

1. Let G be a finite group. Use symmetric functions to prove that there exists a finite normal extension $F \subseteq K$ that has $G \cong G(K/F)$.

2. Consider the polynomial $x^3 - 4x^2 + 6x - 2$ in $\mathbb{Q}[x]$. Let $\alpha_1, \alpha_2, \alpha_3$ be the three roots of the polynomial in $\mathbb{C}$.

   a. Find the minimal polynomial of $\alpha_1 + \alpha_2 + \alpha_3$ over $\mathbb{Q}$.

   b. Find a cubic polynomial over $\mathbb{Q}$ that has α_1^2, α_2^2 and α_3^2 as its roots.

3. For each group below, two normal series are given. Find isomorphic refinements of these two series,

   a. For the group $\mathbb{Z}$, the two series are

$$\langle 0 \rangle \leqslant 60\mathbb{Z} \leqslant 20\mathbb{Z} \leqslant \mathbb{Z}$$

$$\langle 0 \rangle \leqslant 245\mathbb{Z} \leqslant 49\mathbb{Z} \leqslant \mathbb{Z}$$

   b. For the group $\mathbb{Z}/24\mathbb{Z}$, the two series are

$$\langle 0 \rangle \leqslant \langle 3 \rangle \leqslant \mathbb{Z}/24\mathbb{Z}$$

$$\langle 0 \rangle \leqslant \langle 8 \rangle \leqslant \mathbb{Z}/24\mathbb{Z}$$

4. Find all composition series for $\mathbb{Z}/48\mathbb{Z}$, and prove that they're all isomorphic by describing the successive quotients of each series.

5. In a permutation group S_n, a *transposition*, also called a 2-*cycle*, is a permutation of the form $(i\ j)$.

   a. Prove that any $\sigma \in S_n$ can be expressed as a product of transpositions.

   b. In S_5, find three different ways to express (12345) as a product of transpositions.

   c. Let $\sigma \in S_n$ and suppose we have expressed σ as a product of transpositions in two different ways

$$\sigma = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_n$$

$$\sigma = \rho_1 \circ \rho_2 \circ \cdots \circ \rho_m$$

   Prove that $n \equiv m \pmod 2$. In other words, among all the ways to express σ as a product of transpositions, either they all involve an odd number of transpositions, or they all involve an even number of transpositions. Depending on this parity, we either say σ is an *even permutation* or an *odd permutation.*

   d. Prove that the set of all even permutations forms a subgroup of S_n. It is called the *alternating group* A_n.

   e. Prove that A_5 is a simple group.

Bonus. Explain why S_4 is a solvable group, but S_5 is not.