MA 362 — Homework 11 — Due Friday, April 19

1. Let $F = \mathbb{Z}/2\mathbb{Z}$, let $f(x) = x^2 + x + 1 \in F[x]$, and let K be the splitting field of $f(x)$.

   a. Prove that $G(K/F)$ is a solvable group.

   b. Prove that K is an extension by radicals.

2. Prove that any polynomial in $\mathbb{Q}[x]$ of the form $f(x) = ax^6 + bx^3 + c$ is solvable by radicals by explicitly giving an extension of $\mathbb{Q}$ (in terms of radical expressions involving a, b and c) that contains the roots of $f(x)$.

3. Prove that if G is a solvable group and K is a normal subgroup of G, then both K and G/K are solvable groups. You can use any of the isomorphism theorems.

4. a. Prove that S_4 is a solvable group by constructing a composition series with abelian quotients.

   b. For any field F, prove that if $f(x) \in F[x]$ is a polynomial of degree $\leqslant 4$ with splitting field K then $G(K/F)$ is isomoprhic to a subgroup of S_4. Conclude that polynomials of degree 4 have solvable Galois groups.

5. Let H be a subgroup of S_5. Prove that if H contains a transposition τ and a 5-cycle σ, then $H = S_5$ (This will be useful to us later).

Bonus. Suppose $f(x) \in \mathbb{Q}[x]$ has exactly three real roots and two complex roots, and let K be its splitting field. Prove that $G(K/\mathbb{Q})$ contains a 2-cycle and a 5-cycle (thinking of elements in the Galois group as permutations of the five roots).