MA 362 — Homework 7 — Due Friday, March 8

1. For F a field, define the *formal derivative*

$$D : F[x] \longrightarrow F[x]$$

to be the algebraic analog of the derivative from calculus:

$$D\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=1}^{n} i a_i x^{i-1}.$$

   i. For $f(x), g(x) \in F[x]$, prove that $D(f(x) + g(x)) = D(f(x)) + D(g(x))$.
   ii. For $f(x), g(x) \in F[x]$, prove that $D(f(x)g(x)) = D(f(x))g(x) + f(x)D(g(x))$.
   iii. Let α in $\overline{F}$ be a root of some $f(x) \in F[x]$. Prove that α has multiplicity $m > 1$ if and only if α is a root of $D(f(x))$
   iv. Use the formal derivative to give a proof (different from the proof in class) that if F is a field with characteristic zero, then F is perfect.

2. For each field extension of $\mathbb{Q}$ given below, find a primitive element $\alpha \in E$ so that $E = \mathbb{Q}(\alpha)$.
   i. $E = \mathbb{Q}(\sqrt{2}, \sqrt{5})$
   ii. $E = \mathbb{Q}(i, \sqrt[3]{2})$
   iii. $E = \mathbb{Q}(\sqrt[4]{2}, \sqrt[6]{2})$

3. Prove or disprove:
   i. If F is an algebraically closed field, then F is perfect.
   ii. If $F \subseteq E$ is a finite separable extension, then $|G(E/F)| = [E : F]$
   iii. If F is a perfect field and $F \subseteq E$ is a finite extension, then E is perfect.

4. Let $F = \mathbb{Z}/2\mathbb{Z}$ be the field with two elements, and consider $f(x) = x^3 + x + 1$ in $F[x]$.
   i. Prove that $f(x)$ is irreducible over F
   ii. The field extension $F[x]/(f(x)) = F(\alpha)$ has exactly eight elements. Write the multiplication table for this finite field.
   iii. Since finite fields are perfect, $F(\alpha)$ is a splitting field for $f(x)$. What are the other two roots of $f(x)$ (in terms of α)?
   iv. Describe the set of all automorphisms $G(F(\alpha)/F)$, and write the multiplication table for this group.

Bonus. Recall that for any finite extension $F \subseteq E$, the index $\{E : F\}$ divides the degree $[E : F]$. In class we said that E is a *separable* extension of F if in fact $\{E : F\} = [E : F]$. Giving a name to the opposite extreme, we say that E is *totally inseparable* over F if $\{E : F\} = 1$. Similarly, an element $\alpha \in E$ is called *totally inseparable* over F if the simple extension $F(\alpha)$ is totally inseparable over F.

Prove that E is totally inseparable over F if and only if for all $\alpha \in E$, α is totally inseparable over F.