HW3.

1. $x^8 - 1$ factors as $(x-1)(x+1)(x^2+1)(x^4+1)$.

$x-1$ and $x+1$ have roots in $\mathbb{Q}$.

x^2+1 has roots $\pm i \in \mathbb{C}$.

x^4+1 has roots $\pm \frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2} \in \mathbb{C}$.

So the splitting field of x^8-1 is

$E = \mathbb{Q}\left(i, -i, \frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)$.

I claim that $E = \mathbb{Q}(i, \sqrt{2})$:

proof: certainly $E \subseteq \mathbb{Q}(\sqrt{2}, i)$, since all 6 elements adjoined to create E are elements of $\mathbb{Q}(\sqrt{2}, i)$.

we know $i \in \mathbb{Q}(\sqrt{2}, i)$ is also in E. that will show if we prove $\sqrt{2} \in E$, that $\mathbb{Q}(\sqrt{2}, i) \subseteq E$, hence $\mathbb{Q}(\sqrt{2}, i) = E$.

But note that $\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)+\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)$
$= 2\frac{\sqrt{2}}{2} = \sqrt{2}$ is a sum of elements of E, so $\sqrt{2} \in E$.

So $\mathbb{Q}(\sqrt{2}, i)$ is ~~a~~ ~~subfi~~ the splitting field of x^8-1, with

$[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}] = \underbrace{[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}(\sqrt{2})]}_{=2}\underbrace{[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]}_{=2} = 4$

since $i \notin \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{R}$ is a root of x^2+1.

, since min poly of $\sqrt{2}$ is x^2-2.

$$0 = (a + b\alpha + c\alpha^2)^3 - 3$$
$$= a^3 + 3a^2b\alpha + 3ab^2\alpha^2 + b^3\alpha^3 + 3a^2c\alpha^2 + 6abc\alpha^3$$
$$+ 3b^2c\alpha^4 + 3ac^2\alpha^4 + 3bc^2\alpha^5 + c^3\alpha^6$$

But $\alpha^3 = 2$, $\alpha^4 = 2\alpha$, $\alpha^5 = 2\alpha^2$, $\alpha^6 = 4$

$$= a^3 + 3a^2b\alpha + 3ab^2\alpha^2 + 2b^3 + 3a^2c\alpha^2 + 12abc$$
$$+ \cancel{3}\, 6b^2c\alpha + 6ac^2\alpha + 6bc^2\alpha^2 + 4c^3$$

$$0 = \left(a^3 + 2b^3 + 12abc + 4c^3\right)$$
$$+ \left(3a^2b + 6b^2c + 6ac^2\right)\alpha$$
$$+ \left(3ab^2 + 3a^2c + 6bc^2\right)\alpha^2$$

$\Rightarrow$ all 3 coeffs are 0.
solving this system (I used mathematica) shows that the only solution is $a = b = c = 0$.

So $x^3 - 3$ has no root in $\mathbb{Q}(\sqrt[3]{2})$, so it is irreducible over this field. So

$$[\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3}) : \mathbb{Q}(\sqrt[3]{2})] = 3,$$

and so $[\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3}) : \mathbb{Q}] = 9$. If $\alpha = \sqrt[3]{2}$, $\beta = \sqrt[3]{3}$, a basis is given by:

$$\{1, \alpha, \alpha^2, \beta, \alpha\beta, \alpha^2\beta, \beta^2, \alpha\beta^2, \alpha^2\beta^2\}$$

c. $\mathbb{C} = \mathbb{R}(i)$, and i has minimal polynomial x^2+1 of degree 2, so

$$[\mathbb{C} : \mathbb{R}] = 2$$

d. $[\mathbb{R} : \mathbb{Q}]$ is infinite.

any finite extension of $\mathbb{Q}$ is a finite $\mathbb{Q}$-vector space, which is countable. But $\mathbb{R}$ is uncountable.

e. $[\mathbb{R}(\sqrt{3}) : \mathbb{R}] = 1$ since $\mathbb{R}(\sqrt{3}) = \mathbb{R}$.

3. $\Rightarrow$. If L is algebraic over F, then since E is a subfield of L, we know E is algebraic over F. And for any $\alpha \in L$, α is a root of some $f(x) \in F[x]$, and since $F \subseteq E$ we can think of $f(x)$ as a polynomial in $E[x]$ too. So L is algebraic over E.

$\Leftarrow$ Pick $\alpha \in L$ and let $e(x) = \sum_{i=0}^{n} \beta_i x^i \in E[x]$ be a polynomial with $e(\alpha) = 0$.

Then each β_i, $0 \le i \le n$ is algebraic over F.

so the extension $F \subseteq F(\beta_0, \beta_1, \ldots, \beta_n)$ is algebraic over F.

And since α is a root of a polynomial in $F(\beta_0, \ldots, \beta_n)$, we know that $F(\beta_0, \ldots, \beta_n, \alpha)$ is an algebraic extension of $F(\beta_0, \ldots, \beta_n)$.

not needed.

But $[F(\beta_0 \ldots \beta_n) : F] \le \prod_{i=0}^{n} \deg(\text{min poly of } \beta_i)$.

and $[F(\beta_0, \ldots, \beta_n, \alpha) : F(\beta_0 \ldots \beta_n)] \le n$.

so the extension $F(\beta_0 \ldots \beta_n, \alpha)$ is a finite extension of F, which means it's an algebraic extension. So α is algebraic over F. $\square$.

4. a.

Suppose α has minimal polynomial

$$f(x) = \sum_{i=0}^{n} a_i x^i$$ where n is odd.

Then $f(\alpha) = \sum_{i=0}^{n} a_i \alpha^i = 0$

$$\Rightarrow \sum_{\substack{i=0 \\ i \text{ even}}}^{n} a_i \alpha^i + \sum_{\substack{i=0 \\ i \text{ odd}}}^{n} a_i \alpha^i = 0$$

$$\Rightarrow \sum_{\substack{i=0 \\ i \text{ even}}}^{n} a_i (\alpha^2)^{i/2} + \sum_{\substack{i=0 \\ i \text{ odd}}}^{n} a_i (\alpha^2)^{\frac{i-1}{2}} \cdot \alpha = 0$$

$$\Rightarrow \left(\sum_{\substack{i=0 \\ i \text{ even}}}^{n} a_i (\alpha^2)^{i/2} \right) + \left(\sum_{\substack{i=0 \\ i \text{ odd}}}^{n} a_i (\alpha^2)^{\frac{i-1}{2}} \right) \alpha = 0$$

$\Rightarrow$ α is a root of a linear polynomial in $F(\alpha^2)$

$$\Rightarrow \quad [F(\alpha) : F(\alpha^2)] = 1$$

$\Rightarrow$ $[F(\alpha^2) : F] = [F(\alpha) : F]$ is odd,

so the minimal polynomial of α^2 over F has odd degree.

b. $\mathbb{R} \subseteq \mathbb{R}(i)$, i has minimal polynomial x^2+1,

and since $i^2 = -1 \in \mathbb{R}$ we see that

$\mathbb{R}(i^2) \subsetneq \mathbb{R}(i)$.

5 a Let ~~f~~ $f(x) \in \overline{F}_E[x]$. Since E ~~is~~ is algebraically closed, we know E contains a root of $f(x)$, call it α. Consider:

~~Suppose $\alpha \notin \overline{F}_E$ then we'd have~~

$F \subseteq \overline{F}_E \subseteq \overline{F}_E(\alpha)$, a tower of algebraic extensions. by Q3 on this homework, this means $\overline{F}_E(\alpha)$ is algebraic over F. So α is algebraic over F, so $\alpha \in \overline{F}_E$ by definition.

Since every polynomial in $\overline{F}_E[x]$ has a root in $\overline{F}_E$, $\overline{F}_E$ is algebraically closed.

b. $\pi \in \mathbb{C}$ is transcendental over $\mathbb{Q}$, so it is not in $\overline{\mathbb{Q}}_{\mathbb{C}}$.

(you can also use a countable/uncountable argument here).