Math 362 HW 2.

1. (a) A basis for E over F is $\{1, \alpha, \alpha^2, \alpha^3\}$.

Note: $\alpha^4 - \alpha^3 + \alpha^2 - \alpha + 1 = 0$

(b)

	1	α	α^2	α^3
1	1	α	α^2	α^3
α	α	α^2	α^3	$\alpha^3 - \alpha^2 + \alpha - 1$
α^2	α^2	α^3	$\alpha^3 - \alpha^2 + \alpha - 1$	-1
α^3	α^3	$\alpha^3 - \alpha^2 + \alpha - 1$	-1	$-\alpha$

(c) $\alpha^5 = -1$

binom thm:

$$(\alpha^2 + \alpha^3)^5 = \alpha^{10} + 5\alpha^{11} + 10\alpha^{12} + 10\alpha^{13} + 5\alpha^{14} + \alpha^{15}$$

Note: $\alpha^{10} = 1$

$$= 1 + 5\alpha + 10\alpha^2 + 10\alpha^3 + \underbrace{5\alpha^4} + (-1)$$

$$= 1 + 5\alpha + 10\alpha^2 + 10\alpha^3 + 5(\alpha^3 - \alpha^2 + \alpha - 1) - 1$$

$$= -5 + 10\alpha + 5\alpha^2 + 15\alpha^3.$$

$$(\alpha^3 + \alpha^4)(1 + \alpha^3) = \alpha^3 + \alpha^6 + \alpha^4 + \alpha^7$$

$$= \alpha^3 - \alpha + (\alpha^3 - \alpha^2 + \alpha - 1) + (-\alpha^2)$$

$$= -1 - 2\alpha^2 + 2\alpha^3.$$

2. Since $[E:F]=4$, we know that for any $\beta \in E$,

β is algebraic, and $F \subseteq F(\beta) \subseteq E \Rightarrow [F(\beta):F] \mid 4$.

but $[F(\beta):F]$ is the degree of the minimal polynomial of β.

so any $\beta \in E$ is the root of a polynomial of degree dividing 4, which means β will also be the root of a (maybe not irreducible) monic polynomial of degree 4, after multiplying its minimal polynomial by something of the right degree. So, assume we have a deg 4 poly and find the coefficients.

a.

$$(\alpha^2)^4 + a(\alpha^2)^3 + b(\alpha^2)^2 + c(\alpha^2) + d = 0$$

$$\Rightarrow -\alpha^3 + a(-\alpha) + b(\alpha^3 - \alpha^2 + \alpha - 1) + c\alpha^2 + d = 0$$

$$\Rightarrow (b-1)\alpha^3 + (c-b)\alpha^2 + (-a+b)\alpha + (d-b) = 0.$$

But $\{1, \alpha, \alpha^2, \alpha^3\}$ is a basis. So

$$\begin{aligned} b-1 &= 0 & & b=1 \\ c-b &= 0 & \Rightarrow\ & c=1 \\ -a+b &= 0 & & a=0 \\ d-b &= 0. & & d=1. \end{aligned}$$

So the minimal polynomial of α^2 will divide

$x^4 + x^3 + x^2 + x + 1$. But this is irreducible,

so it's the min poly of α^2.

b.

$$(1+\alpha^3)^4 + a(1+\alpha^3)^3 + b(1+\alpha^3)^2 + c(1+\alpha^3) + d = 0.$$

$$\Rightarrow (1 + 4\alpha^3 + 6\alpha^6 + 4\alpha^9 + \alpha^{12}) +$$
$$+ a(1 + 3\alpha^3 + 3\alpha^6 + \alpha^9)$$
$$+ b(1 + 2\alpha^3 + \alpha^6)$$
$$+ c(1 + \alpha^3)$$
$$+ d \qquad = 0$$

$$\Rightarrow 1 + 4\alpha^3 + 6(-\alpha) + 4(-\alpha^3 + \alpha^2 - \alpha + 1) + \alpha^2$$
$$+ a + 3a\alpha^3 + 3a(-\alpha) + a(-\alpha^3 + \alpha^2 - \alpha + 1)$$
$$+ b + 2b\alpha^3 + b(-\alpha)$$
$$+ c + c\alpha^3$$
$$+ d \quad = 0$$

$$\Rightarrow (2a + 2b + c)\alpha^3 + (4 + 1 + a)\alpha^2$$
$$+ (-6 - 4 - 3a - a - b)\alpha + (1 + 4 + 2a + b + c + d) = 0.$$

Setting coefficients equal to 0 and solving gives:

$x^4 - 5x^3 + 10x^2 - 10x + 5$, which is irreducible.

$$(\alpha+\alpha^2)^4 + a(\alpha+\alpha^2)^3 + b(\alpha+\alpha^2)^2 + c(\alpha+\alpha^2) + d = 0$$

$$\Rightarrow \alpha^4 + 4\alpha^3(\alpha^2) + 6\alpha^2(\alpha^2)^2 + 4\alpha(\alpha^2)^3 + (\alpha^2)^4$$

$$+ a(\alpha^3 + 3\alpha^2(\alpha^2) + 3\alpha(\alpha^2)^2 + (\alpha^2)^3)$$

$$+ b(\alpha^2 + 2\alpha(\alpha^2) + (\alpha^2)^2)$$

$$+ c(\alpha + \alpha^2)$$

$$+ d = 0.$$

$$\Rightarrow (\alpha^3 - \alpha^2 + \alpha - 1) + 4(-1) + 6(-\alpha) + 4(-\alpha^2) + (-\alpha^3)$$

$$+ a\alpha^3 + 3a(\alpha^3 - \alpha^2 + \alpha - 1) + 3a(-1) + a(-\alpha)$$

$$+ b\alpha^2 + 2b\alpha^3 + b(\alpha^3 - \alpha^2 + \alpha - 1)$$

$$+ c\alpha + c\alpha^2$$

$$+ d = 0$$

$$\Rightarrow (4a+3b)\alpha^3 + (-3a+c-5)\alpha^2 + (2a+b+c-5)\alpha$$

$$+ (-6a - b + d - 5) = 0$$

solving for a, b, c, d gives $a=0$, $b=0$, $c=5$, $d=5$

So $x^4 + 5x + 5$, which is irreducible.

3. a. Let $\alpha = \sqrt{2}$.

then $F = \mathbb{Q}(\sqrt{2})$ is isomorphic to $\dfrac{\mathbb{Q}[x]}{x^2-2}$,

(min poly of $\sqrt{2}$)

so it is two dimensional as a $\mathbb{Q}$-vector space

b. Let $\beta = \sqrt{2} + \sqrt{3}$.

then $\beta^2 = 5 + 2\sqrt{6}$

$\Rightarrow \beta^2 - 5 = 2\sqrt{6}$

$\Rightarrow \beta^4 - 10\beta^2 + 25 = 24$

$\Rightarrow \beta^4 - 10\beta^2 + 1 = 0$. And $x^4 - 10x^2 + 1$ is irreducible,

so $[\mathbb{Q}(\sqrt{2}+\sqrt{3}) : \mathbb{Q}] = 4$.

c. ~~Using degrees~~: $\cancel{\mathbb{Q} \subseteq \mathbb{Q}(\beta) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})}$

$\cancel{\Rightarrow [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] =}$

its enough to prove $\sqrt{2} \in \mathbb{Q}(\beta)$, $\sqrt{3} \in \mathbb{Q}(\beta)$.

First, $\sqrt{6} = \frac{\beta^2 - 5}{2} \in \mathbb{Q}(\beta)$.

then $\sqrt{6}\,\beta = \sqrt{6}(\sqrt{2}+\sqrt{3}) = 2\sqrt{3} + 3\sqrt{2} \in \mathbb{Q}(\beta)$

so $\sqrt{6}\,\beta - 2\beta = \sqrt{2} \in \mathbb{Q}(\beta)$.

and $\beta - \sqrt{2} = \sqrt{3} \in \mathbb{Q}(\beta)$.

so $\mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathbb{Q}(\beta)$.

So theyre equal.

4. a. Let $\beta = x^2 + (f(x))$.

Then $\beta^2 = x^4 + (f(x)) = 2 + (f(x))$,

since $x^4 - 2 \in (f(x))$.

b. Since $[L : \mathbb{Q}] = 4$ and $[\mathbb{Q}(\beta) : \mathbb{Q}] = 2$,

we know $[L : \mathbb{Q}(\beta)] = 2$, so they aren't equal.

c. the min poly of α is $x^4 - 2$.

d. over $\mathbb{Q}(\beta)$, $x^4 - 2$ factors as

$(x^2 - \sqrt{2})(x^2 + \sqrt{2})$, and of these,

α is a root of $x^2 - \sqrt{2}$.

5. Pick any $\alpha \in E$, $\alpha \notin F$. Then we have a tower $F \subseteq F(\alpha) \subseteq E$,

and since $F(\alpha)$ is a proper extension of F, the degree $[F(\alpha) : F] > 1$. But then

$p = [E : F] = [E : F(\alpha)]\,[F(\alpha) : F]$ is prime,

so $[E : F(\alpha)] = 1$, so $E = F(\alpha)$.