MA 362 — Homework 9 — Due Friday, April 5

1. Consider the field $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$

   i. Prove that K is a finite normal extension of $\mathbb{Q}$ (this should be quick.)

   ii. Use the main theorem of Galois theory to compute the following numbers. (Recall that for an intermediate field E, we use $\lambda(E)$ to mean the subgroup of $G(K/\mathbb{Q})$ that fixes E.)

      a. $\{K : \mathbb{Q}\}$
      b. $|G(K/\mathbb{Q})|$
      c. $|\lambda(\mathbb{Q})|$
      d. $|\lambda(\mathbb{Q}(\sqrt{2}, \sqrt{3}))|$
      e. $|\lambda(\mathbb{Q}(\sqrt{6}))|$
      f. $|\lambda(\mathbb{Q}(\sqrt{30}))|$
      g. $|\lambda(\mathbb{Q}(\sqrt{2} + \sqrt{5}))|$
      h. $|\lambda(K)|$

2. Give an example of two fields K_1, K_2 that are both finite normal extensions of $\mathbb{Q}$, so that there is *no* isomorphism $K_1 \longrightarrow K_2$, but there is an isomorphism $G(K_1/\mathbb{Q}) \longrightarrow G(K_2/\mathbb{Q})$.

3. A finite normal extension $F \subseteq K$ is called *abelian* if the group $G(K/F)$ is an abelian group. Prove that when this is the case, every intermediate field $F \subseteq E \subseteq K$ is a finite normal extension of F, and furthermore that every such field is also an abelian extension of F.

4. Let F be a field, and let $K = F(y_1, y_2, y_3)$ be a transcendental extension. As discussed in class, the elementary symmetric functions are

$$\begin{aligned} s_1 &= y_1 + y_2 + y_3 \\ s_2 &= y_1y_2 + y_1y_3 + y_2y_3 \\ s_3 &= y_1y_2y_3 \end{aligned}$$

   Soon we'll prove that every symmetric function can be expressed as a rational function of the elementary symmetric functions, and that $E = F(s_1, s_2, s_3) \subseteq K$ is a finite normal extension with Galois group $G(K/E) \cong S_3$.

   i. Express $y_1^3 + y_2^3 + y_3^3$ as a rational function of the elementary symmetric functions.

   ii. The subgroup generated by $(123) \in G(K/E)$ is a normal subgroup of this group. Let L be the fixed field of this subgroup. Describe L as an extension of E. There should only be one nontrivial automorphism $\sigma \in G(L/E)$. Describe this automorphism.

Bonus. With notation as in problem 4, but this time with four transcendental elements, we let $K = F(y_1, \dots, y_4)$ and let $E = F(s_1, \dots, s_4)$. Find a field L satisfying $E \subseteq L \subseteq K$ so that the Galois group $G(K/L)$ is isomorphic to the dihedral group D_8 (the 8-element group of the symmetries of a square.)