MA 362 — Homework 4 — Due Friday, Feb 8

The first two questions on this homework don't have much to do with the field extensions. Instead, the first asks you to fill in some details of a claim I've often made in class: if F is a field, then every ideal in $F[x]$ is generated by a single element. The second will give you some practice with chains in partially ordered sets as they relate to a property of rings that we've seen before.

1. let F be a field and let $I \subseteq F[x]$ be an ideal. Prove that $I = (f(x))$ for some polynomial $f(x) \in F[x]$ by doing the following:

   a. Prove it in the two trivial cases: when I does not contain any nonzero polynomials, and when I contains a nonzero constant polynomial.

   b. Now if I does contain a nonzero polynomial, and contains no constant polynomials, then let $f(x)$ be a nonzero polynomial in I of smallest degree and prove that $I = (f(x))$. You will need the division algorithm for polynomial rings.

2. A commutative ring R is called *noetherian* if every ideal I in R is finitely generated. Prove the following two equivalent definitions of this property:

   a. R is noetherian if and only if for any chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ there exists an index N such that $I_N = I_{N+1} = I_{N+2} = \cdots$. This characterization is sometimes called "no infinite ascending chains".

   b. R is noetherian if and only if every nonempty set S of ideals in R has a maximal element. (in more detail, S is a partially ordered set with respect to $\subseteq$. A maximal element of S is an ideal J satisfying that if $J \subseteq I$ for some $I \in S$ then $J = I$.)

3. Consider the field extension $\mathbb{Q} \subseteq E = \mathbb{Q}(\sqrt{2})$, and recall that $\{1, \sqrt{2}\}$ is a $\mathbb{Q}$-basis for E. This means we can define a $\mathbb{Q}$-linear map from E to itself just by defining the map's values on the basis element.

   Consider the linear map $\psi : E \to E$ defined by $\psi(1) = 1$ and $\psi(\sqrt{2}) = -\sqrt{2}$.

   a. Prove directly that ψ is a field isomorphism. (An isomorphism from an object to itself to called an *automorphism*).

   b. Prove that for any $r \in \mathbb{Q} \subseteq E$, $\psi(r) = r$. We say that ψ *fixes* the subfield $\mathbb{Q}$.

   c. Prove that there are only two automorphisms of E that fix $\mathbb{Q}$, namely the ψ from this problem and the identity function. **Hint:** Prove that if ξ is such an isomorphism, then $\xi(1) = 1$ and $\xi(\sqrt{2}) = \pm\sqrt{2}$.

4. The real number $\alpha = \sqrt{1 + \sqrt{2}}$ is algebraic over $\mathbb{Q}$. A real number β is called a *conjugate* of α if β and α have the same minimal polynomial. Find all conjugates of α.

5. Let $F \subseteq E$ be a field extension, and let G be the set of all automorphisms of E that fix F. Prove that G is a group under composition. Describe this group when $\mathbb{Q} \subseteq E$ is the field extension from question 3.

Bonus. Consider the field extension $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, which has $\mathbb{Q}$-basis $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$. Thinking of E as a vector space, any linear transformation $E \to E$ can be written as a 4×4 matrix with entries in $\mathbb{Q}$. For example, the linear function σ defined by $\sigma(1) = \sqrt{2} + \sqrt{3}$, $\sigma(\sqrt{2}) = 1 + \sqrt{6}$, $\sigma(\sqrt{3}) = 2 + 2\sqrt{3}$, $\sigma(\sqrt{6}) = 2\sqrt{2} + \sqrt{3}$ would be represented by the matrix:

$$M_\sigma = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

To use this matrix to find out where the element $\alpha = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$ gets sent by σ, we write our element as a column vector:

$$c_\alpha = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

and do the matrix multiplication $M_\sigma \cdot c_\alpha$, then translate the resulting column vector back into an element of E.

Find the set of all matrices that represent automorphisms of E that fix $\mathbb{Q}$, and find a generating set for this matrix group.