HW 9

1. $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ is seperable because $\mathbb{Q}$ is perfect, and K is the splitting field of $\{x^2-2,\ x^2-3,\ x^2-5\} \subseteq \mathbb{Q}[x]$. So K is a finite normal extension of $\mathbb{Q}$.

It is a degree 8 extension, so

$$[K:\mathbb{Q}] = \{K:\mathbb{Q}\} = |G(K/\mathbb{Q})| = 8.$$ so

a. $\{K:\mathbb{Q}\} = 8$

b. $|G(K/\mathbb{Q})| = 8$

c. $|\lambda(\mathbb{Q})| = |\{\sigma \in \overset{G(K/\mathbb{Q})}{\overset{\shortparallel}{G}} : \sigma \text{ fixes } \mathbb{Q}\}| = |G(K/\mathbb{Q})| = 8$

(Recall: $\lambda(E) = G(K/E)$, for any $\mathbb{Q} \subseteq E \subseteq K$.)

d.
$$\begin{array}{l} K \\ \big| \leftarrow \text{deg } 2 \\ \mathbb{Q}(\sqrt{2},\sqrt{3}) \\ \big| \leftarrow \text{deg } 4 \\ \mathbb{Q} \end{array}$$

$\Rightarrow$ $|\lambda(\mathbb{Q}(\sqrt{2},\sqrt{3}))|$

$= |G(K/\mathbb{Q}(\sqrt{2},\sqrt{3}))|$

$= [K:\mathbb{Q}(\sqrt{2},\sqrt{3})] = 2$

(since $\mathbb{Q}(\sqrt{2},\sqrt{3}) \subseteq K$ is a finite normal extension.

e.
$$\begin{array}{l} K \\ \big| \leftarrow \text{deg } 4 \\ \mathbb{Q}(\sqrt{6}) \\ \big| \leftarrow \text{deg } 2 \\ \mathbb{Q} \end{array}$$

Similarly: $|\lambda(\mathbb{Q}(\sqrt{6}))| = 4$

f.
$$\begin{tikzcd} K \arrow[d, dash, "\leftarrow \text{deg } 4"] \\ \mathbb{Q}(\sqrt{30}) \arrow[d, dash, "\leftarrow \text{deg } 2"] \\ \mathbb{Q} \end{tikzcd} \qquad |\lambda(\mathbb{Q}(\sqrt{30}))| = 4$$

g. The min. poly of $\sqrt{2}+\sqrt{5}$ is $x^4 - 14x^2 + 9$,

so
$$\begin{tikzcd} K \arrow[d, dash, "\leftarrow \text{deg } 2"] \\ \mathbb{Q}(\sqrt{2}+\sqrt{5}) \arrow[d, dash, "\leftarrow \text{deg } 4"] \\ \mathbb{Q} \end{tikzcd} \qquad |\lambda(\mathbb{Q}(\sqrt{2}+\sqrt{5}))| = 2$$

h. $|\lambda(K)| = |G(K/K)| = 1$. only the identity automorphism fixes K.

2. $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ are both separable ($\mathbb{Q}$ is perfect) and both splitting fields (of x^2-2 or x^2-3, resp.), and $|G(\mathbb{Q}(\sqrt{2})/\mathbb{Q})| = |G(\mathbb{Q}(\sqrt{3})/\mathbb{Q})| = 2$.

Since 2 is prime, the only group of size 2 is the cyclic group of order 2. So these Galois groups are isomorphic

But the fields are not isomorphic: in $\mathbb{Q}(\sqrt{2})$, the polynomial x^2-2 splits. But in $\mathbb{Q}(\sqrt{3})$ it doesn't.

3. If $F \subseteq K$ is an abelian extension, then consider a field E with $F \subseteq E \subseteq K$.

$F \subseteq E$ is a finite normal extension iff $G(K/E)$ (MTGT) is a <u>normal</u> subgroup of $G(K/F)$. But since $G(K/F)$ is abelian, all of its subgroups are normal. So $F \subseteq E$ is a finite normal extension.

$G(E/F)$ is the quotient $G(K/F) / G(K/E)$ (MTGT). Since quotients of abelian groups are abelian, $F \subseteq E$ is an abelian extension.

4a. Note: $s_1^3 = y_1^3 + y_2^3 + y_3^3 + 3(y_1y_2^2 + y_1^2y_2 + y_1y_3^2 + y_1^2y_3 + y_2y_3^2 + y_2^2y_3)$
$+ 6(y_1y_2y_3)$.

And $s_1s_2 = (y_1y_2^2 + y_1^2y_2 + y_1y_3^2 + y_1^2y_3 + y_2y_3^2 + y_2^2y_3) + (3y_1y_2y_3)$

So $y_1^3 + y_2^3 + y_3^3 = s_1^3 - 3s_1s_2 + 3s_3$.

b. First we need to find an expression involving y_1, y_2, y_3 that is fixed by $\langle (123) \rangle$ but not by the rest of S_3. Note that $T = y_1y_2^2 + y_1^2y_3 + y_2y_3^3$ is such an expression.

So we've found:

$$E = F(s_1, s_2, s_3) \subseteq F(s_1, s_2, s_3, T) \subseteq F(y_1, y_2, y_3).$$

Is this the fixed field of $\langle (123) \rangle$?

By MTGT,

$$\begin{tikzcd} F(y_1,y_2,y_3) \arrow[d, dash, "\deg 3"] \arrow[dd, bend left=60, dash, "\deg 6"] \\ \text{fixed field of } \langle(123)\rangle \arrow[d, dash, "\text{must be } 2"'] \\ F(s_1,s_2,s_3) \end{tikzcd}.$$

So if T has minimal polynomial over F of degree 2, then we're done. Look at:

$$(X - (y_1y_2^2 + y_1^2y_3 + y_2y_3^2))(X - (y_1^2y_2 + y_1y_3^2 + y_2^2y_3))$$

$$= X^2 - (y_1y_2^2 + y_1^2y_2 + y_1y_3^2 + y_1^2y_3 + y_2y_3^2 + y_2^2y_3)X$$

$$\cancel{+ (y_1^3y_2^2 + y_1^3y_2y_3 + y_1y_2^4y_3 +}$$

$$+ (y_1^3y_2^3 + y_1^3y_3^3 + y_2^3y_3^3 + y_1^4y_2y_3 + y_1y_2^4y_3 + y_1y_2y_3^4 + 3y_1^2y_2^2y_3^2)$$

$$= X^2 - (s_1s_2 - 3s_3)X + (9s_3^2 + s_2^3 + s_1^3s_3 - 6s_1s_2s_3)$$

$$\in F(s_1,s_2,s_3)[X].$$

So $F(s_1,s_2,s_3)(T)$ is a degree 2 extension of $F(s_1,s_2,s_3)$, so it is the fixed field of $\langle(123)\rangle$. The nontrivial automorphism is defined by sending $T = y_1y_2^2 + y_1^2y_3 + y_2y_3^3$ to its conjugate, which is $y_1^2y_2 + y_1y_3^2 + y_2^2y_3$