MA 362 — Homework 3 — Due Friday, Feb 1

1. Consider the polynomial $x^8 - 1 \in \mathbb{Q}[x]$. Find a finite extension field $\mathbb{Q} \subseteq E$ where this polynomial completely factors into a product of 8 linear polynomials in $E[x]$, but where $x^8 - 1$ does not completely factor into linear polynomials using any proper subextension of E. This field E is called the *splitting field* of $x^8 - 1$ What is the degree $[E : \mathbb{Q}]$?

2. Find the degree of each field extension listed below. If the degree is finite, give a basis for the extension field over the subfield.

   a. $[\mathbb{Q}[x]/(f(x)) : \mathbb{Q}]$ where $f(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$ of degree d.
   b. $\left[\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{6}, \sqrt[3]{24}) : \mathbb{Q}\right]$
   c. $[\mathbb{C} : \mathbb{R}]$
   d. $[\mathbb{R} : \mathbb{Q}]$
   e. $\left[\mathbb{R}(\sqrt{3}) : \mathbb{R}\right]$

3. Let $F \subseteq E \subseteq L$ be a tower of field extensions. Prove that L is algebraic over F if and only if L is algebraic over E and E is algebraic over F (Don't assume the field extensions are finite degree).

4. Let $F \subseteq E$ be a field extension, and let $\alpha \in E$ be element which is algebraic over F.

   a. Prove that if the minimal polynomial of α over F has odd degree, then α^2 also has minimal polynomial over F of odd degree, and $F(\alpha) = F(\alpha^2)$.
   b. Give a concrete example of fields $F \subseteq E$ and $\alpha \in E$ where the minimal polynomial of α has even degree and $F(\alpha) \neq F(\alpha^2)$.

5. Let $F \subseteq E$ be a field extension and assume that E is algebraically closed.

   a. Prove that $\overline{F}_E$, the algebraic closure of F in E, is also algebraically closed.
   b. In the special case $\mathbb{Q} \subseteq \mathbb{C}$, prove that $\overline{\mathbb{Q}}_{\mathbb{C}}$, the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, is not equal to $\mathbb{C}$ (you can assume the complex numbers are an algebraically closed field).

Bonus. Let $F \subseteq E$ be a field extension. A finite set of elements $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq E$ is called *algebraically dependent* over F if there exists a nonzero polynomial in n variables $f(x_1, x_2, \ldots, x_n) \in F[x_1, x_2, \ldots, x_n]$ so that $f(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$. If no such polynomial exists, then the set is called *algebraically independent*.

   a. Prove that a set with one element $\{\alpha\}$ is algebraically independent if and only if α is transcendental over F.
   b. Prove that if the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq E$ contains one algebraic element, then the set is algebraically dependent.
   c. Prove that $\{\pi^2 + \pi + 1, \pi^3 - \sqrt{\pi} - 1\}$ is algebraically dependent over $\mathbb{Q}$.
   d. Determine whether $\{\pi, e\}$ is an algebraically independent set over $\mathbb{Q}$ (Note: nobody in the world knows the answer to this question).