MA 362 — Homework 1 — Due Friday, Jan 18

1. Let F be a field. In class we proved that every polynomial $f(x) \in F[x]$ has an irreducible factor, which is a much weaker version of the following statement(s): Prove that

   a. Any polynomial $f(x) \in F[x]$ with $\deg(f(x)) \geqslant 1$ can be written as a product of irreducible polynomials.

   b. The factorization from part a. is unique up to re-ordering the factors and multiplying by a constant. In other words, if

   $$f(x) = p_1(x)p_2(x)\dots p_n(x)$$

   and

   $$f(x) = q_1(x)q_2(x)\dots q_m(x)$$

   are two different ways to write $f(x)$ as a product of irreducible polynomials, then $n = m$ and each $p_i(x)$ is equal to some $q_j(x)$, possibly after multiplying by a unit in F (a "unit in F" is the same thing as a non-zero constant polynomial in $F[x]$).

2. For each number below, find a polynomial in $\mathbb{Q}[x]$ having that number as a root, and determine whether or not the polynomial you found is *irreducible*.

   a. $\alpha = \sqrt{11}$

   b. $\beta = \sqrt{3} + \sqrt{5}$

   c. $\gamma = \sqrt[3]{7}$

   c. $\delta = \sqrt[3]{8}$

3. Consider the complex number $\zeta = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. We can observe that $\zeta^6 = 1$, which means that ζ is a root of $x^6 - 1 \in \mathbb{Q}[x]$. But this polynomial is reducible. Find an *irreducible* polynomial that has ζ as a root.

4. Kronecker's Theorem tells us that $E = \mathbb{Q}[x]/(x^2 - 2)$ is a field extension of $\mathbb{Q}$ that contains a root of $x^2 - 2$. But it also meets the conditions in definition 30.1 to be a *vector space* over $\mathbb{Q}$. What is the dimension of this vector space? Give a basis.

5. Give an example of an algebraic field extension of $\mathbb{Q}$ which is not a subfield of $\mathbb{R}$

Bonus. The polynomial $f(x) = x^4 - 7x^2 + 10$ has $\sqrt{2}$ as a root. This means that the algebraic extension of $\mathbb{Q}$ given by $\mathbb{Q}[x]/(x^2 - 2)$ contains a root of $f(x)$. Construct an algebraic extension of $\mathbb{Q}$ in which $f(x)$ has *four* roots. This is called thet *splitting field* of $f(x)$.