MA 362 — Homework 8 — Due Friday, March 22

1. This was proven in class on Friday, March 8, but the proof was a little rushed, and it's good to refresh after Spring break:

   Let K be a finite normal extension of F, and let E be an intermediate field $F \subseteq E \subseteq K$. Prove that

   i. K is a finite normal extension of E.

   ii. The subgroup of $G(K/F)$ consisting of the automorphisms of K that fix E is exactly $G(K/E)$.

   iii. If σ, τ are elements of $G(K/F)$, then they induce the same isomorphism on E if and only if they're in the same left coset of $G(K/E)$. In other words:

$$\sigma\Big|_E = \tau\Big|_E \Longleftrightarrow \tau G(K/E) = \sigma G(K/E).$$

2. Find the splitting field K of $x^4 - x^2 - 2$ over $\mathbb{Q}$. Describe the group $G(K/\mathbb{Q})$. Write the lattice of subgroups, and find the fixed field of each subgroup.

3. Repeat the above problem for the polynomial $x^7 - 1$ over $\mathbb{Q}$.

4. Let K be the splitting field of $x^3 - 2$ over $\mathbb{Q}$ and let L be the splitting field of $x^3 - 1$ over $\mathbb{Q}$. For each field, describe the group of automorphisms of that field fixing $\mathbb{Q}$, and explain how and why these two fields are different. If you were asked about the splitting field of $x^3 - 5$ and its group of automorphisms, would you expect it to look like $G(K/\mathbb{Q})$, or like $G(K/\mathbb{Q})$, or like a completely different group?

Bonus. A finite normal extension $F \subseteq K$ is called *cyclic* if $G(K/F)$ is a cyclic group.

   i. Let $F \subseteq E \subseteq K$ be a tower of field extensions, and suppose K is a finite normal extension of F and that E is a finite normal extension of F. Prove that if K is cyclic over F, then K is cyclic over E and E is cyclic over F

   ii. Prove that if K is cyclic over F, then for each divisor d of $[K : F]$, there is exactly one intermediate field $F \subseteq E \subseteq K$ with $[E : F] = d$.