

Topological Complexity of Graphic Arrangements

Nathan Fieldsteel

ABSTRACT. By combining Yuzvinsky's criteria from [13] with tools from graph theory, we obtain an explicit combinatorial condition on a finite graph G which guarantees that the higher topological complexity TC_s of the complement of the associated graphic arrangement \mathcal{A}_G is equal to the dimensional upper bound $sr - 1$, where r is the rank of \mathcal{A}_G .

1. Introduction

The topological complexity $TC(X)$ of a space X is an integer which measures the extent to which any motion planning algorithm for X must be discontinuous. $TC(X)$ is a special case of the Schwarz genus introduced in [12]. Specifically, $TC(X)$ is the Schwarz genus of the path fibration

$$\pi : X^I \rightarrow X \times X \text{ given by } \pi(\gamma) \mapsto (\gamma(0), \gamma(1)).$$

The exact value of the topological complexity of a space can be difficult to compute, but its value is of interest for spaces like configuration spaces and their generalizations; spaces for which explicit motion planning algorithms are often desired for practical applications. Examples of such spaces include the space of configurations of a mechanical system, or the space of configurations of a multi-body system in a 2 or 3 dimensional space. Higher topological complexities, denoted $TC_s(X)$ for $s > 2$, were defined in [11] to address algorithms for planning more complicated motions. Recent work has been done on computing the topological complexity of hyperplane arrangement complements and other combinatorially determined spaces, for example [1],[4],[5],[6],[14] and [13]. In this paper we focus on a particular class of hyperplane arrangements called graphic arrangements.

In recent work in [13], Yuzvinsky gives a combinatorial condition on a complex hyperplane arrangement \mathcal{A} which guarantees that the topological complexity of the arrangement complement is maximized. An arrangement satisfying the condition defined in [13] is called *large*. In the case of graphic arrangements, we show that this condition is equivalent to a strengthened version of the inequality in a theorem of Nash-Williams which guarantees that the edges of a graph can be decomposed into two acyclic subgraphs. Our main result is the following.

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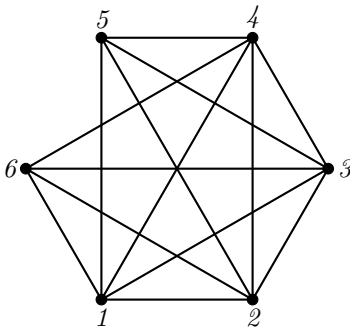
THEOREM 1.1. *Let $G = (V, E)$ be a graph with $|V| = r + 1$ and no isolated vertices, and let \mathcal{A}_G be its associated graphic arrangement. Then \mathcal{A}_G is large if and only if G contains a spanning subgraph H having $2r - 1$ edges and satisfying that for every nonempty, non-singleton subset $U \subseteq V$ we have*

$$|E_H(U)| < 2(|U| - 1).$$

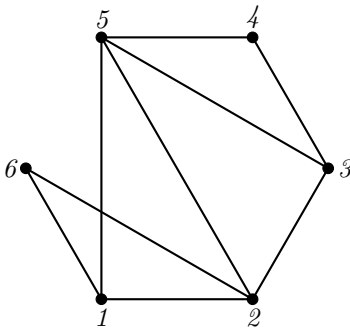
Here $E_H(U)$ denotes the set of edges of the subgraph of H induced by U . In particular, if such an H exists, then the higher topological complexity TC_s of the complement of \mathcal{A}_G is equal to $sr - 1$.

Before proceeding, we give a quick illustrative example.

EXAMPLE 1.2. *Let \mathcal{A} be the arrangement attained by removing one hyperplane from the arrangement A_5 . In other words, \mathcal{A} is the arrangement in \mathbb{C}^6 consisting of the hyperplanes defined by the 14 equations $\{x_1 - x_2, x_1 - x_3, x_1 - x_4, x_1 - x_5, x_1 - x_6, x_2 - x_3, x_2 - x_4, x_2 - x_5, x_2 - x_6, x_3 - x_4, x_3 - x_5, x_3 - x_6, x_4 - x_5, x_4 - x_6\}$. This arrangement is the graphic arrangement associated to the graph attained by deleting one edge from the complete graph K_6 .*



It is shown in [13] that the topological complexity of the complement of the A_5 arrangement is $5s - 1$. The topological complexity $TC_s(\mathcal{A})$ of the complement of the arrangement attained by deleting one hyperplane is at most $5s - 1$, but it may be lower. If we can find a full rank subarrangement \mathcal{A}' with $TC_s(\mathcal{A}') = 5s - 1$, then we can conclude that $TC_s(\mathcal{A}) = 5s - 1$ as well. Such a subarrangement would correspond to a subgraph H which satisfies the theorem above. Let \mathcal{A}' be the subarrangement determined by the subgraph H shown below.



It is easy to confirm that for any non-empty, non-singleton subset U of the vertex set of H , the subgraph induced by U has strictly fewer than $2(|U| - 1)$ edges. Applying the theorem above lets us conclude that $TC_s(\mathcal{A}) = TC_s(\mathcal{A}') = 5s - 1$.

2. Hyperplane Arrangements

We begin by establishing the terminology and results we will need for our discussion of hyperplane arrangements. Additional background and details can be found in [10].

DEFINITION 2.1. A hyperplane arrangement \mathcal{A} is a finite set $\{H_1, \dots, H_n\}$ of codimension 1 linear subspaces of a complex affine space \mathbb{C}^r .

An arrangement \mathcal{A} is called *central* if the intersection $H_1 \cap H_2 \cap \dots \cap H_n$ is nonempty, and *essential* if the intersection contains exactly one point. When we refer to the *combinatorics* of the arrangement, we mean the partially ordered set of all intersections of subsets of \mathcal{A} , ordered by reverse inclusion. This is called the *intersection lattice* of \mathcal{A} . When we refer to the *topology* of the arrangement, we mean the topology of its complement

$$M_{\mathcal{A}} = \mathbb{C}^r \setminus \bigcup_{i=1}^n H_i.$$

A subset $\{H_{i_1}, \dots, H_{i_t}\}$ of t hyperplanes of \mathcal{A} is called *independent* if the intersection $H_{i_1} \cap H_{i_2} \cap \dots \cap H_{i_t}$ has codimension t , and is called *dependent* otherwise. It turns out that the cohomology of $M_{\mathcal{A}}$ is determined by the combinatorial data of the arrangement.

THEOREM 2.2 (Orlik, Solomon [9]). Let $E_{\mathcal{A}}$ be the exterior algebra with generators $\{e_1, \dots, e_n\}$ in natural correspondence with the hyperplanes in \mathcal{A} , and let $I_{\mathcal{A}}$ be the ideal in $E_{\mathcal{A}}$ given by

$$\left\langle \sum_{j=1}^t (-1)^j e_{i_1} \wedge \dots \wedge \hat{e}_{i_j} \wedge \dots \wedge e_{i_t} \mid \{H_{i_1}, \dots, H_{i_t}\} \text{ is a dependent set in } \mathcal{A} \right\rangle.$$

$I_{\mathcal{A}}$ is called the Orlik-Solomon ideal of \mathcal{A} , and the Orlik-Solomon algebra of \mathcal{A} , denoted $A(\mathcal{A})$, is the quotient of $E_{\mathcal{A}}$ by $I_{\mathcal{A}}$. The Orlik-Solomon algebra is isomorphic to the cohomology of the complement: $H^*(M_{\mathcal{A}}, \mathbb{C}) \cong A(\mathcal{A})$.

Any linear ordering \preceq of the hyperplanes in \mathcal{A} determines a basis for $A(\mathcal{A})$, called the *no-broken-circuit basis*. A *circuit* in \mathcal{A} is a minimal dependent set of hyperplanes, and a *broken circuit* is a circuit with its minimal (with respect to \preceq) hyperplane removed. A subset of \mathcal{A} is called *no-broken-circuit* or **nbc** if it does not contain a broken circuit.

The monomials in $E_{\mathcal{A}}$ are naturally identified with subsets of \mathcal{A} . For any choice of linear ordering \preceq , the images of the **nbc** monomials in $A(\mathcal{A})$ form a \mathbb{C} -basis for the Orlik-Solomon algebra, called the **nbc** basis for that ordering.

3. Topological Complexity and Motion Planning

Let X be a topological space and suppose we are interested in the motion planning problem for X [3]: given any two points a and b in X we would like a path $\gamma : I \rightarrow X$ starting at a and ending at b . We would like this assignment of

paths to be continuous in a and b , but this is only possible when X is a contractible space. So instead we seek a collection of local assignments of paths to pairs. If we let $\pi : X^I \rightarrow X \times X$ be the *path fibration* of X , defined by $\pi(\gamma) = (\gamma(0), \gamma(1))$, we are led to the following definition, which first appeared in [2].

DEFINITION 3.1. *A motion planning algorithm for X , or simply a motion planner is a finite open cover $\{U_0, \dots, U_n\}$ of $X \times X$ together with a map $s_i : U_i \rightarrow X^I$ satisfying that $\pi \circ s_i = \text{id}_{U_i}$. The open sets U_i are called the local domains of the motion planner.*

If a pair (a, b) is in one local domain, while a nearby pair (a', b') is in a different local domain, then it is possible that the motion planner will assign very different paths to these pairs, even if the pairs themselves are very close. For this reason, in practical applications it is desirable to have a motion planner with as few local domains as possible. The extent to which this goal can be achieved is measured by the topological complexity of X .

DEFINITION 3.2. *The topological complexity of X , denoted by $TC(X)$, is the smallest integer n such that there exists a motion planner for X with $n + 1$ local domains $\{U_0, \dots, U_n\}$.*

Note that we are using the reduced version of topological complexity; a space for which a motion planner exists with a single local domain would have topological complexity 0. Higher topological complexity, generalizing the notions given above, was defined in [11]. In the above definitions, replace $X \times X$ with the s -fold product of X , and replace the path fibration with

$$\pi_s : X^I \rightarrow \underbrace{X \times X \times \dots \times X}_{s \text{ times}}$$

defined by evaluation at s points in I :

$$\pi_s(\gamma) := (\gamma(0), \gamma(\frac{1}{s-1}), \gamma(\frac{2}{s-1}), \dots, \gamma(\frac{s-2}{s-1}), \gamma(1))$$

Higher topological complexity $TC_s(X)$ is one less than the number of open subsets needed to cover the base $X \times X \times \dots \times X$ so that on each open subset, π_s admits a continuous section. When $s = 2$, this recovers the definition of TC given above. Before we proceed, we state a few useful properties of TC_s which we will use in later sections.

PROPOSITION 3.3 ([13]).

- (1) $TC_s(X)$ is an invariant of the homotopy type of X [2].
- (2) If X has homotopy dimension r , $TC_s(X) \leq sr$.
- (3) In the case where X is the complement of an arrangement of hyperplanes in \mathbb{C}^r , we have that $TC_s(X) \leq sr - 1$.
- (4) There is a lower bound for $TC_s(X)$, given by the higher zero-divisors-cup-length:

$$\text{zcl}_s(X) \leq TC_s(X).$$

This lower bound $\text{zcl}_s(X)$ is called the s^{th} zero-divisors-cup-length of X . It is computed in terms of the multiplication structure of $H^*(X; \mathbb{C})$. More precisely, if we let K_s denote the kernel of the cup product map

$$\underbrace{H^*(X; \mathbb{C}) \otimes H^*(X; \mathbb{C}) \otimes \dots \otimes H^*(X; \mathbb{C})}_{s \text{ times}} \rightarrow H^*(X; \mathbb{C}),$$

then the s^{th} zero-divisors cup length of X , denoted $\text{zcl}_s(X)$, is the largest integer z so that the ideal K_s^z is non-zero.

4. Large Arrangements

For general spaces, it can be very difficult to compute the exact value of $TC_s(X)$. The main technique for proving that a space has a certain topological complexity is to explicitly construct a motion planner for it having a number of local domains greater by one than the cohomological lower bound. This is extremely difficult for complicated spaces. Of course, if the cohomological lower bound is equal to the dimensional upper bound, then the value of $TC_s(X)$ is also equal to that upper bound.

In what follows, \mathcal{A} will denote an essential arrangement of complex hyperplanes in \mathbb{C}^r . By abuse of notation, we will interchangeably use \mathcal{A} to refer to the hyperplane arrangement or its complement. In [13], Yuzvinsky gives a combinatorial condition on \mathcal{A} which guarantees that the cohomological lower bound and the dimensional upper bound for $TC_s(\mathcal{A})$ are equal. We recall the relevant definitions here.

DEFINITION 4.1. *A pair (B, C) of subsets of \mathcal{A} is called a basic pair if there exists a linear order \preceq on the set of hyperplanes such that the following conditions are met:*

- (1) B and C are disjoint.
- (2) B is maximal **nbc** (meaning that $|B| = r$) for the order \preceq .
- (3) C is **nbc** for the order \preceq .

An arrangement \mathcal{A} is called large if it admits a basic pair with $|C| = r - 1$.

Our result for graphic arrangements is proven by leveraging the result in [13] which states that large arrangements have maximal topological complexity.

PROPOSITION 4.2 ([13]). *If \mathcal{A} contains a basic pair with $|C| = r - 1$ then $\text{zcl}_s(X) = sr - 1$ and hence $TC_s(X) = sr - 1$.*

5. Graphic Arrangements

The condition in Proposition 4.2, when satisfied, gives the topological complexity of any essential hyperplane arrangement. We are interested in applying it to the case of graphic arrangements, a class of arrangements which has overlap with reflection arrangements and Coxeter arrangements, and which has very well-behaved combinatorics.

DEFINITION 5.1. *Let $G = (V, E)$ be a finite graph with vertices $\{v_1, \dots, v_{r+1}\}$. Consider a complex affine space \mathbb{C}^{r+1} with coordinates $\{x_1, \dots, x_{r+1}\}$, and for any edge $(v_i, v_j) \in E$ let H_{ij} denote the hyperplane in \mathbb{C}^{r+1} defined by $x_i - x_j = 0$. The graphic arrangement associated to G is given by $\{H_{ij} | (v_i, v_j) \in E\}$.*

If G is a connected graph, then the intersection of all of the H_e is 1-dimensional. By projecting to the orthogonal complement of this subspace, we see that the complement of \mathcal{A} is homotopy equivalent to the complement of an essential arrangement in \mathbb{C}^r with the same combinatorics as \mathcal{A}_G . Since TC_s is a homotopy invariant, the

topological complexity of this essential arrangement will be the same as the topological complexity of \mathcal{A}_G .

5.1. When is a graphic arrangement large? The topological complexity $TC_s(\mathcal{A}_G)$ is at most $sr - 1$, with equality guaranteed when \mathcal{A}_G is a large arrangement. So we would like a combinatorial condition on G which is equivalent to the condition that \mathcal{A}_G is large. To find such a condition, we should first formulate Definition 4.1 for the graphic arrangement case.

A set of hyperplanes in \mathcal{A}_G is independent if and only if the corresponding subset S of the edge set of G is acyclic. Furthermore, for a given linear order \preceq on E , an independent set of hyperplanes in \mathcal{A}_G is **nbc** if and only if the corresponding subset S of the edge set of G is acyclic and for any path P in S and any edge e not in S such that $\{e\} \cup P$ is a cycle, e is not the minimal element of $\{e\} \cup P$ with respect to \preceq . For brevity, we will use **nbc** to refer to subsets of the edge set of G which correspond to **nbc** subsets of \mathcal{A}_G .

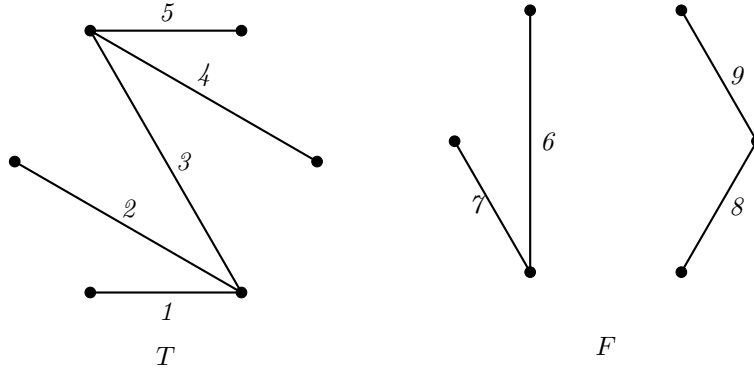
Using this, we can rephrase the condition that \mathcal{A}_G is large in terms of the underlying graph as follows

PROPOSITION 5.2. *When G is a connected graph, there exists a basic pair (B, C) in \mathcal{A}_G if and only if there exists a pair (T, F) of disjoint subsets of E , and a linear ordering \preceq on E , satisfying the following.*

- (1) T is a spanning tree for G .
- (2) F is a disjoint union of at least two trees.
- (3) If P is a path in T and e is an edge in F such that $P \cup \{e\}$ forms a cycle, then e is not the minimal element of that cycle with respect to \preceq .
- (4) If P is a path in F and e is an edge in T such that $P \cup \{e\}$ forms a cycle, then e is not the minimal element of that cycle with respect to \preceq .

It is immediate that \mathcal{A}_G is a large arrangement if and only if there exists a pair (T, F) and an order \preceq as above with $|F| = r - 1$. In this situation, it must be the case that F is a disjoint union of exactly two trees, which together form a spanning subgraph of G .

EXAMPLE 5.3. *Revisiting Example 1.2, if we decompose the edge set of the subgraph G' into subsets T and F and choose an ordering on the edges as shown below.*



This partition (T, F) of the edge set and ordering of the edges corresponds to a basic pair (B, C) for the graphic arrangement \mathcal{A}' with $|C| = r - 1 = 4$. The

existence of this pair is what guarantees that $TC_s(\mathcal{A}') = 5s - 1$. It is worth noting that it will not generally be true that the linear ordering is such that all edges in T are less than all edges in F .

Proposition 5.2 is a reformulation of Definition 4.1 for the graphic arrangement case. In the above example, we simply found the decomposition and ordering by hand. But using tools from graph theory, we can obtain a simpler condition on the graph G which is equivalent to the conditions in Proposition 5.2 being satisfied.

6. Arboricity and a theorem of Nash-Williams

We've seen that questions about whether a graphic arrangement is large are closely related to questions about decomposing finite graph into acyclic subsets, and this is a well-understood subject. For a finite graph $G = (V, E)$, there is a smallest integer k such that E can be written as a disjoint union of k acyclic subsets. This number is called the *arboricity* of G and is tightly connected to the density of edges among the vertex-induced subgraphs of G . In particular, a theorem of Nash-Williams gives us the following.

THEOREM 6.1 (Nash-Williams [8]). *Let $G = (V, E)$ be a finite graph. For any subset $U \subseteq V$, we will use $E(U)$ to denote the edge set of the subgraph induced by U . The edge set of G can be partitioned into k forests if and only if for all nonempty $U \subseteq V$ we have*

$$|E(U)| \leq k(|U| - 1).$$

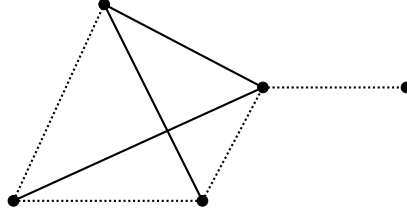
This inequality is a condition on the density of the edges of G among the vertex-induced subgraphs of G . If the edge set of G is a union of a small number of acyclic subsets, then no vertex-induced subgraph of G can have a high density of edges, and in fact the reverse is true.

Applying this theorem with $k = 2$ to graph with $r + 1$ vertices and $2r - 1$ edges, we see that when

$$|E(U)| \leq 2(|U| - 1)$$

for all $U \subseteq V$, not only will the edge set of G be partitioned into two forests, but in fact one of the two forests will be a spanning tree and the other will be a disjoint union of exactly two trees. This additional structure is nothing but numerics; it can't be the case that both of the forests have fewer than r edges, since G has a total of $2r - 1$ edges. And neither forest can have more than r edges, since such a subgraph would have a cycle. So the only possibility is that one of the forests has exactly r edges and is a spanning tree, and the other has $r - 1$ edges and is a forest composed of exactly two trees. These two subsets are naturally the pair (T, F) which form a candidate for a basic pair for the arrangement \mathcal{A}_G . But we don't yet know whether there exists the necessary linear order \preceq on the edge set. For an example of why G having arboricity 2 is not sufficient to guarantee that \mathcal{A}_G is large, consider the complete graph K_4 with one additional vertex and pendant edge shown below. For all subsets of its vertex set, it satisfies the Nash-Williams inequality, so it can be partitioned into a spanning tree T (dotted edges) and a forest F composed of exactly two trees (solid edges). Note that one of the trees in F is just an isolated vertex.

However, T and F cannot form a basic pair. To see this, let \preceq be any linear ordering on the edges of this graph. If we restrict our attention to the K_4 subgraph,



we see that every dotted edge can be completed to a cycle by a path of solid edges. And every solid edge can be completed to a cycle by a path of dotted edges. This means that no matter which edge in the K_4 is minimal with respect to \preceq , it will force one of either T or F to fail to be **nbc**.

If we let U be the vertex set of the K_4 , we see that it satisfied the Nash-Williams inequality with equality, i.e. $6 = |E(U)| = 2(|U| - 1) = 2(4 - 1)$. In what follows, we will show that such subsets are the only obstruction to the existence of a linear ordering \preceq with the needed properties.

7. Result

Our main result is that strengthening the inequality in the Nash-Williams theorem guarantees the existence of the desired linear order, and so guarantees that \mathcal{A}_G is a large arrangement.

THEOREM 7.1. *Let $G = (V, E)$ be a graph with $|V| = r + 1$ and no isolated vertices, and let \mathcal{A}_G be its associated graphic arrangement. Then \mathcal{A}_G is large if and only if G contains a spanning subgraph H having $2r - 1$ edges and satisfying that for every nonempty, non-singleton $U \subseteq V$ we have*

$$|E_H(U)| < 2(|U| - 1),$$

where $E_H(U)$ denotes the set of edges of the subgraph of H induced by U . In particular, if this inequality is satisfied, then the higher topological complexity TC_s of the complement of \mathcal{A}_G is equal to $sr - 1$.

In order to prove this Theorem, we will make use of the following technical lemma. This is a translation of theorem 4.1 from [13] into the language of graphs .

LEMMA 7.2. *Let $H = (V, E)$ be a graph with $|E_H(U)| < 2(|U| - 1)$ for all nonempty, non-singleton $U \subseteq V$, and suppose that E can be written as a union of disjoint subsets T and F , where T is a spanning tree and F is a proper forest. Then there exists a linear ordering \preceq of E so that*

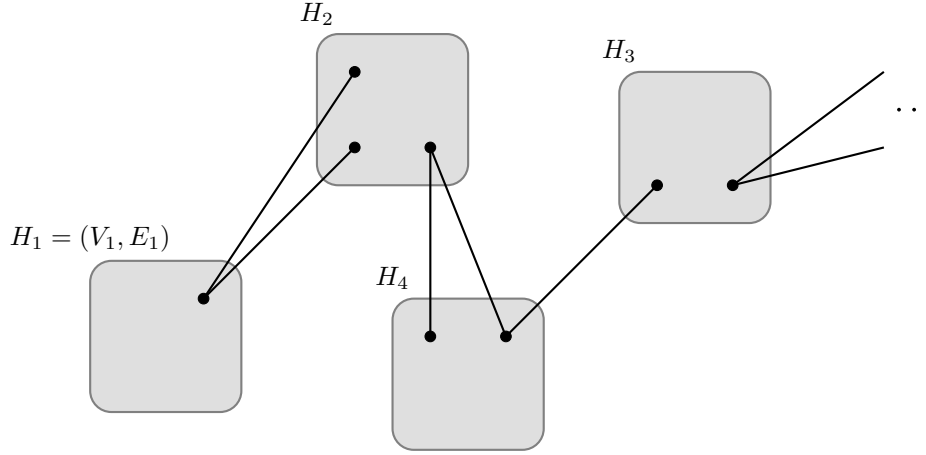
- (1) *If P is a path in T and e is an edge in F such that $P \cup \{e\}$ forms a cycle, then e is not the minimal element of that cycle with respect to \preceq .*
- (2) *If P is a path in F and e is an edge in T such that $P \cup \{e\}$ forms a cycle, then e is not the minimal element of that cycle with respect to \preceq .*

PROOF. We will prove this lemma by induction on $|V|$. It is vacuously true when $|V| = 1$.

If $|V| > 1$, then the disjoint trees in F partition V into disjoint subsets V_1, \dots, V_k with $k \geq 2$. Let E_i denote $E_H(V_i)$. Since T is a spanning tree, there must be at least one edge connecting a vertex in V_i to a vertex in V_j for some $i \neq j$. We let E_0 denote the set of all such edges, so that

$$E_0 = E \setminus \bigcup_{i=1}^k E_i,$$

and E is the disjoint union $E_0 \cup E_1 \cup \dots \cup E_k$. We will denote by H_i the graph (V_i, E_i) . In this way, H can be seen as a disjoint union of at least 2 vertex-induced subgraphs H_1, \dots, H_k , represented here as grey boxes, connected by the edges of E_0 .



We will construct the linear ordering \preccurlyeq on E by giving an order on each E_i , then concatenating these orderings so that when $i < j$, all edges in E_i are less than all edges in E_j .

First, let e be an edge in E_0 and let P be any path in F . Since e connects two disjoint subtrees of F , $\{e\} \cup P$ does not form a cycle. For this reason, the edges of E_0 can be chosen to be minimal among the edges of E without introducing any broken circuits. The ordering of E_0 itself can be chosen arbitrarily.

Now consider E_i for $i \geq 1$. The graph H_i satisfies that $|E_{H_i}(U)| < 2(|U| - 1)$ for all nonempty, non-singleton $U \subseteq V_i$. $F \cap E_i$ is a spanning tree for H_i by design, and $T \cap E_i$ is a forest. We see immediately that $T \cap E_i$ must be a proper forest, because if it were a spanning tree, then we would have $|E_{H_i}(V_i)| = 2(|V_i| - 1)$, a contradiction. Since H_i has strictly fewer vertices than H , we know by induction that there is a linear ordering \preccurlyeq on E_i so that the following two conditions are met.

- (1) If P is a path in $F \cap E_i$ and e is an edge in $T \cap E_i$ such that $P \cup \{e\}$ forms a cycle, then e is not the minimal element of that cycle with respect to \preccurlyeq .
- (2) If P is a path in $T \cap E_i$ and e is an edge in $F \cap E_i$ such that $P \cup \{e\}$ forms a cycle, then e is not the minimal element of that cycle with respect to \preccurlyeq .

Let \preccurlyeq be the linear ordering of E defined by concatenating the arbitrary ordering of E_0 with the orderings of the E_i as described above. All that remains is to verify that the ordering \preccurlyeq satisfies the necessary conditions.

Let P be a path in T and suppose that e is an edge in F so that $P \cup \{e\}$ is a cycle. If $P \cap E_0$ is nonempty, then e cannot be the minimal element of that cycle by construction. If $P \cap E_0$ is empty, then P is contained in H_i for some $1 \leq i \leq k$, so e must also be an edge in H_i , so by induction e is not the minimal element of the cycle.

Similarly, let P be a path in F and let e be an edge in T so that $P \cup \{e\}$ is a cycle. P must be a path in H_i for some $1 \leq i \leq k$, which means e must be an edge in H_i and so is not the minimal edge of the cycle by induction. \square

With the above lemma in place, we now proceed to the proof of the theorem.

PROOF. Since any arrangement which contains a large subarrangement is itself large [13], it is enough to show that the graphic arrangement \mathcal{A}_H is a large arrangement.

\mathcal{A}_H is large if and only if it contains a basic pair (B, C) , which is equivalent to the existence of a pair (T, F) and a linear order \preceq as described in Proposition 5.2.

Suppose $|E(U)| < 2(|U| - 1)$ for all $U \subseteq V$. By Nash-Williams, we know that the edge set of H can be written as a disjoint union of two forests T and F . As mentioned above, we can assume that T is a spanning tree and $|F| = r - 1$, and so F is a disjoint union of exactly two trees. Since F is a proper forest, the above lemma guarantees the existence of a linear order \preceq on E so that (T, F) forms a basic pair, so \mathcal{A}_H is a large arrangement and $TC_s(\mathcal{A}_H) = sr - 1$.

For the reverse implication, Let G be a graph satisfying that for each spanning subgraph H with $2r - 1$ edges, there is at least one nonempty, non-singleton subset $U \subseteq V$ for which the above strict inequality does not hold. We will show that the edges of H cannot form a basic pair, and so G cannot be large. If H is such a subgraph and U satisfies $|E_H(U)| > 2(|U| - 1)$, then by Nash-Williams the edges of H can't be decomposed into two acyclic subsets and so cannot correspond to a basic pair in \mathcal{A}_G . Next suppose H is a subgraph so that $|E_H(U)| \leq 2(|U| - 1)$ for all nonempty, non-singleton subsets U , but with at least one subset U satisfying $|E_H(U)| = 2(|U| - 1)$. We will show that there cannot exist a linear ordering on the edges of H satisfying the needed conditions. Let H' be the subgraph of H induced by U . By Nash-Williams, the edge set of H' will decompose into a disjoint union of two forests. Since $|H'| = 2(|U| - 1)$, both of these forests must be spanning trees of H' , call them T_1 and T_2 . For an arbitrary edge e in T_1 , there exists a path in T_2 which is completed to a cycle by e . So e cannot be the minimal edge of H' . By symmetry, we can make the same argument about an arbitrary edge in T_2 . So no linear ordering of the edges of H' can satisfy the conditions of Proposition 5.2, and so the edges of H cannot form a basic pair for \mathcal{A}_G . \square

8. Applications

We close with some examples of graphs that determine large graphic arrangements.

EXAMPLE 8.1. *Let \mathcal{A} be the graphic arrangement associated to the complete tripartite graph $K_{r-1,1,1}$. The arrangement \mathcal{A} is large and $TC_s(\mathcal{A}) = sr - 1$.*

PROOF. Let v_1 and v_2 be the vertices of the singleton parts, and let U be a nonempty, non-singleton subset of the vertex set. If neither v_1 nor v_2 are in U , then $|E(U)| = 0 < 2(|U| - 1)$. If exactly one of the v_1 or v_2 is in U , then $|E(U)| = |U| - 1 < 2(|U| - 1)$. If both v_1 and v_2 are in U , then $|E(U)| = 2(|U| - 2) < 2(|U| - 1)$. So \mathcal{A}_G is large. \square

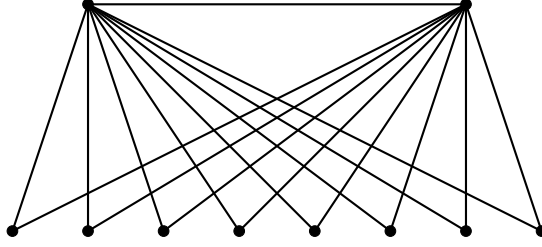


FIGURE 1. The arrangement \mathcal{A}_G associated to the complete tripartite graph $G = K_{8,1,1}$ has $TC_s(\mathcal{A}_G) = 9s - 1$.

EXAMPLE 8.2. Let G be the wheel graph W_{r+1} , that is the graph join of the singleton graph with an r -cycle. The graphic arrangement \mathcal{A}_G is large and $TC(\mathcal{A}_G) = sr - 1$.

PROOF. Let v denote the central vertex of G and let H be the subgraph obtained by deleting one edge which is not incident to v . Let U be a nonempty, non-singleton subset of the vertex set. If v is not in U , then $|E_H(U)| \leq |U| - 1 < 2(|U| - 1)$. If v is in U , then $|E_H(U)| \leq (|U| - 2) + (|U| - 1) = 2|U| - 3 < 2(|U| - 1)$. So the edges of H form a basic pair in \mathcal{A}_G . \square

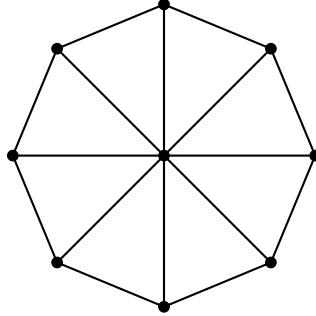


FIGURE 2. The arrangement \mathcal{A}_G associated to the wheel graph $G = W_9$ has $TC_s(\mathcal{A}_G) = 8s - 1$.

EXAMPLE 8.3. Let G be any graph with r vertices for which \mathcal{A}_G is large, and let v_1 and v_2 be two distinct vertices in G . Let G' be the graph formed by adding a new vertex v' to G which is adjacent only to v_1 and v_2 . Then $\mathcal{A}_{G'}$ is large and $TC_s(\mathcal{A}_{G'}) = sr - 1$.

PROOF. Let H be the spanning subgraph of G satisfying theorem 7.1 and let H' be the spanning subgraph of G' formed by adding v' and both its incident edges to H . Let U be a subset of the vertex set of G' . If v' is not in U , then by assumption $|E_{H'}(U)| < 2(|U| - 1)$. If v' is in U , then by assumption $|E_{H'}(U \setminus \{v'\})| < 2(|U| - 2)$, and so $|E_{H'}(U)| = |E_{H'}(U \setminus \{v'\})| + 2 < 2(|U| - 2) + 2 = 2(|U| - 1)$. So the edges of H' will form a basic pair for G' and hence $\mathcal{A}_{G'}$ is large. \square

Since the graphic arrangement associated to K_3 is easily seen to be large, and since both $K_{r-1,1,1}$ and wheel graphs with a deleted edge can be built by iteratively

applying the above construction to K_3 , the above example gives a second proof that the above graphs determined large graphic arrangements.

EXAMPLE 8.4. *Let G be the graph obtained by inserting a diagonal edge into each square of an $n \times 1$ grid graph $P_n \times P_1$ as shown below. Since G can be built by iteratively applying the construction from example 8.3 to K_3 , \mathcal{A}_G is large and hence $TC_s(\mathcal{A}_G) = s(2n - 1) - 1$.*

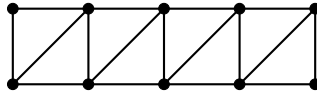


FIGURE 3. Because this 10-vertex graph G can be built by the iterative construction in example 8.3, the associated arrangement has $TC_s(\mathcal{A}_G) = 9s - 1$.

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FIELDSTEEL: MATHEMATICS DEPARTMENT, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, IL 61801, USA

E-mail address: fieldst2@math.uiuc.edu